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INSTANTANEOUS BETHE–SALPETER EQUATION: (SEMI-)ANALYTICAL SOLUTION

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Abstract

The Bethe–Salpeter equation for bound states of a fermion–antifermion pair in the instantaneous approximation for the involved interaction kernel is converted into an equivalent matrix eigenvalue problem with explicitly (algebraically) given matrices.

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1 The Instantaneous Bethe–Salpeter Equation (IBSE)

For a system of massless fermion and antifermion forming bound states with the "pion-like" spin, parity, and charge conjugation quantum numbers $J^{PC} = 0^{-+}$, the (homogeneous) Bethe–Salpeter equation, in free-propagator approximation and instantaneous approximation for the involved interaction kernel, reads for a time-component Lorentz vector interaction (i.e., the Dirac structure $\gamma^0 \otimes \gamma^0$) [1,2]

$$
\begin{aligned}
2\,k\,\Psi_2(k) + \int_0^\infty \frac{\mathrm{d}k'\,k'^2}{(2\pi)^2}\,V_0(k,k')\,\Psi_2(k') &= M\,\Psi_1(k)\ ,\\
2\,k\,\Psi_1(k) + \int_0^\infty \frac{\mathrm{d}k'\,k'^2}{(2\pi)^2}\,V_1(k,k')\,\Psi_1(k') &= M\,\Psi_2(k)\ .
\end{aligned}
\tag{1}
$$

In this set of coupled equations for the two relevant radial Salpeter amplitudes Ψ_1 and Ψ_2 in momentum space, with the bound-state masses M as eigenvalues, the interaction potential $V(r)$, usually formulated in configuration space, enters in form of its standard Fourier–Bessel transforms $V_L(k,k')$, $L = 0, 1$. We adopt a linear potential $V(r) = \lambda\,r$ ($\lambda > 0$) as a simple model for quark confinement.

2 Efficient Method of Solution: Expansion in Terms of Basis States

By insertion of the first of Eqs. (1) into the second and by expansion in terms of sets (distinguished by the angular momenta $\ell = 0, 1$) of basis states for $L_2(R^+)$ —with configuration and momentum-space representations $\phi_i^{(\ell)}(r)$ and $\widetilde{\phi}_i^{(\ell)}(p)$, resp.—the solution of the IBSE (1) reduces to the diagonalization of the matrix[3]

$$\begin{aligned}\mathcal{M}_{ij} = {} & 4\int_0^\infty \mathrm{d}k\, k^4\, \widetilde{\phi}_i^{(0)}(k)\, \widetilde{\phi}_j^{(0)}(k) + 2\int_0^\infty \mathrm{d}k\, k^3\, \widetilde{\phi}_i^{(0)}(k) \int_0^\infty \frac{\mathrm{d}k'\, k'^2}{(2\pi)^2}\, V_0(k,k')\, \widetilde{\phi}_j^{(0)}(k') \\ & + 2\int_0^\infty \mathrm{d}k\, k^2\, \widetilde{\phi}_i^{(0)}(k) \int_0^\infty \frac{\mathrm{d}k'\, k'^3}{(2\pi)^2}\, V_1(k,k')\, \widetilde{\phi}_j^{(0)}(k') \\ & + \int_0^\infty \mathrm{d}k\, k^2\, \widetilde{\phi}_i^{(0)}(k) \int_0^\infty \frac{\mathrm{d}k'\, k'^2}{(2\pi)^2}\, V_1(k,k') \int_0^\infty \frac{\mathrm{d}k''\, k''^2}{(2\pi)^2}\, V_0(k',k'')\, \widetilde{\phi}_j^{(0)}(k'') \ . \qquad (2)\end{aligned}$$

Allowing these basis functions to depend on a variational parameter $\mu > 0$ gives us more freedom in the search for solutions of the IBSE. All integrations in $\mathcal{M}_{ij}$ are evaluated by (truncated) expansions, with the (μ-independent) coefficients[3]

$$\begin{aligned} I_{ij}^{(2)} &\equiv \frac{1}{\mu^2}\int_0^\infty \mathrm{d}k\, k^4\, \widetilde{\phi}_i^{(0)}(k)\, \widetilde{\phi}_j^{(0)}(k)\ , \quad i,j = 0,1,2,\dots\ , \\ b_{ij} &\equiv \frac{1}{\mu}\int_0^\infty \mathrm{d}k\, k^3\, \widetilde{\phi}_i^{(0)}(k)\, \widetilde{\phi}_j^{(0)}(k)\ , \quad k\, \widetilde{\phi}_i^{(0)}(k) = \mu \sum_{j=0}^{N} b_{ji}\, \widetilde{\phi}_j^{(0)}(k)\ , \\ c_{ij} &\equiv \int_0^\infty \mathrm{d}k\, k^2\, \widetilde{\phi}_i^{*(1)}(k)\, \widetilde{\phi}_j^{(0)}(k)\ , \quad \widetilde{\phi}_i^{(0)}(k) = \sum_{j=0}^{N} c_{ji}\, \widetilde{\phi}_j^{(1)}(k)\ , \\ d_{ij} &\equiv \frac{1}{\mu}\int_0^\infty \mathrm{d}k\, k^3\, \widetilde{\phi}_i^{*(1)}(k)\, \widetilde{\phi}_j^{(0)}(k)\ , \quad k\, \widetilde{\phi}_i^{(0)}(k) = \mu \sum_{j=0}^{N} d_{ji}\, \widetilde{\phi}_j^{(1)}(k)\ , \\ V_{ij}^{(\ell)} &\equiv \mu\int_0^\infty \mathrm{d}r\, r^3\, \phi_i^{(\ell)}(r)\, \phi_j^{(\ell)}(r)\ , \quad r\, \phi_i^{(\ell)}(r) = \frac{1}{\mu}\sum_{j=0}^{N} V_{ji}^{(\ell)}\, \phi_j^{(\ell)}(r)\ , \quad \ell = 0,1\ . \end{aligned}$$

The explicit algebraic expressions of all these matrices may be found in Ref. 3.[a]

[a]Let's mention a numerical problem noted for Mathematica 4.0: for, e.g., the matrix element $d_{49,49} = 101/(2\sqrt{689}) = 1.924$ Mathematica finds exactly this value for a working precision of 40 digits but the nonsense value -1.675×10^{22} for the default working precision of 16 digits.

In this way, the IBSE (1) is converted to an eigenvalue problem for the matrix[3]

$$\mathcal{M}_{ij} = 4\,\mu^2\, I^{(2)}_{ij} + 2\,\lambda \sum_{r=0}^{N} b_{ri}\, V^{(0)}_{rj} + 2\,\lambda \sum_{r=0}^{N}\sum_{s=0}^{N} c^*_{ri}\, d_{sj}\, V^{(1)}_{rs} + \frac{\lambda^2}{\mu^2} \sum_{r=0}^{N}\sum_{s=0}^{N}\sum_{t=0}^{N} c^*_{ri}\, c_{st}\, V^{(1)}_{sr}\, V^{(0)}_{tj} \ .$$

3 Analytical Results (for Both Massless and Massive Constituents)

For a matrix size less than or equal to 4, the diagonalization of the matrix $\mathcal{M}_{ij}$ may be even performed analytically. In the one-dimensional case, we find, after minimizing w.r.t. the variational parameter μ, for the lowest bound-state mass[3]

$$M = 4\sqrt{\frac{2\,\lambda}{3\pi}\left(2+\sqrt{5}\right)} \ .$$

For[4] $\lambda = 0.2$ GeV^2, this expression gives $M = 1.696$ GeV, only 2.4 % away from the numerical result $M = 1.656$ GeV, obtained for 15×15 matrices and $N = 49$. For a nonvanishing mass m of the bound-state constituents, we get accordingly

$$M^2 = 8\,m^2 + \frac{8896}{315\,\pi}\,\lambda + \frac{23}{7}\left(\frac{128\,\lambda}{45\,\pi\,m}\right)^2 \qquad (m \neq 0) \ .$$

4 Relations Between Matrix Elements and Accuracy of Expansions

Our final question concerns the errors induced by the necessary truncations of the expansion series. The expansion coefficients b_{ij}, c_{ij}, d_{ij} are not independent but should satisfy (clearly, only in the limit $N \to \infty$ exact) relations of the kind

$$\sum_{r=0}^{N} c^*_{ri}\, c_{rj} = \delta_{ij} \ , \quad \sum_{r=0}^{N} c^*_{ri}\, d_{rj} = \sum_{r=0}^{N} d^*_{ri}\, c_{rj} = b_{ij} \ , \quad \sum_{r=0}^{N} d^*_{ri}\, d_{rj} = I^{(2)}_{ij} \ .$$

For 15×15 matrices and $N = 49$, these relations are fulfilled with relative errors less than 3 %. For comparison, some integrals in (2) may be evaluated exactly.[3]

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